A Study of W_2 - Curvature Tensor of a *N(k)* - Quasi Einstein Manifold

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ABSTRACT

In this paper, the study of W_2 *-curvature tensor in N(k) -quasi Einstein manifold satisfying* $W_2(\xi, X).W_2 = 0$ *is carried out.*

Mathematics subject is classification : 53C25, 53D10, 53D15.

Keywords **:** W_2 *- Curvature tensor, Einstein manifolds, quasi Einstein manifolds, N(k) - quasi Einstein manifolds.*

I. INTRODUCTION

A NON-FLATE n-dimensional Riemannian manifold *(M, g)* is said to be a quasi Einstein manifold [2] it its Ricci tensor satisfies

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y) \quad (1)$$

for all $X, Y \in TM$. Where a and $b \neq 0$ are smooth functions and η be a nonzero 1-form such that

$$g(X, \xi) = \eta(X), \quad \eta(\xi) = 1 \quad (2)$$

for the associated vector field ξ, which is equivalent to

$$Q = aI + b\eta \otimes \xi \quad (3)$$

The 1-form η is called the associated 1-form and the unit vector field ξ is called the generator of the manifold. If the generator ξ belongs to k-nullity distribution *N(k)* then the quasi Einstein manifold is called as an *N(k)*-quasi Einstein manifold [8].

In [8], it was proved that a conformally flat quasi-Einstein manifold is *N(k)*-quasi Einstein. Consequently, it was shown that a 3-dimensional quasi-Einstein manifold is an *N(k)*-quasi - Einstein manifold. The derivation conditions, $R(\xi, X) \cdot R = 0$ and $R(\xi, X) \cdot S = 0$ were also studied, where R and S denote the curvature and Ricci tensor respectively.

On the other hand in [1] the derivative conditions $Z(\xi, X).Z = 0, Z(\xi, X).R = 0$ and $R(\xi, X).Z = 0$ on contact metric manifolds are studied, where Z is concircular curvature tensor. In [7], the condition $X(\xi, X).S = 0$ is studied. In [5], N(k)-quasi Einstein manifolds satisfying the conditions $R(\xi, X).W_2 = 0, W_2(\xi, X).S = 0, P(\xi, X).W_2 = 0$ where P denotes the projective curvature tensor are studied. In this paper, I study the derivation conditions $W_2(\xi, X).W_2 = 0$ on an *N(k)* -quasi Einstein manifold. The paper is organized as follows : Section 2 contains necessary details about *N(k)*-quasi Einstein manifolds and the W_2 curvature tensor. In section 3 the conditions $W_2(\xi, X).W_2 = 0$ on an *N(k)* -quasi Einstein manifold is studied.

II. PRELIMINARIES

Let M be a *(2n + 1)* dimensional Riemannian manifold. The W_2-curvature tensor [4] is defined as

$$W_2(X, Y)Z = R(X, Y)Z - \frac{1}{2n}\left\{\begin{array}{l} g(Y, Z)QX \\ -g(X, Y)QY \end{array}\right\}, \quad (4)$$

$$X, Y, Z \in TM$$

Where R is the Riemannian curvature tensor and

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Q is the Ricci operator defined as

$$S(X,Y) = g(QX,Y), \quad X,Y \in TM \tag{5}$$

Equation (4) can also be written as

$$W_2(X,Y,Z,W) = R(X,Y,Z,W) - \frac{1}{2n}\{g(Y,Z)S(X,W) - g(X,Z)S(Y,W)\} \tag{6}$$

for all $X,Y,Z \in TM$

The k-nullity distribution *N(k)* [6] of a Riemannian manifold *M* is defined by

$$N(k): p \to N_p(k) = \{Z \in T_pM : R(X,Y)Z = k(g(Y,Z)X - g(X,Z)Y)\}$$

for all $X,Y \in TM$, where is some smooth function.

A. *Definition* [8]

The *N(k)* -quasi Einstein manifold is defined as:

Let $(M^{2n+1};g)$ be a quasi Einstein manifold. If the generator ξ belongs to the k-nullity distribution *N(k)* for some smooth function k, then we say that $(M^{2n+1};g)$ is an *N(k)*-quasi Einstein manifold.

B. *Lemma* [3]

In an *(2n+1)*-dimensional *N(k)*- quasi Einstein manifold it follows that

$$k = \frac{a+b}{2n} \tag{7}$$

From equations (1) and (2) it follows that

$$S(X,\xi) = (a+b)\eta(X), \quad X \in TM.$$

$$r = (2n+1)a + b \tag{8}$$

where r is the scalar curvature of M.

Since

$$R(X,Y)\xi = k\{\eta(Y)X - \eta(X)Y\} \tag{9}$$

Using (7) in above equation we get

$$(R(X,Y)\xi = \frac{a+b}{2n}\{\eta(Y)X - \eta(X)Y\} \tag{10}$$

which is equivalent to

$$R(X,\xi)Y = \frac{a+b}{2n}\{\eta(Y)X - g(X,Y)\xi\} = -R(\xi,X)Y \tag{11}$$

Taking X= ξ and Y=X in equation (10) we get

$$R(\xi,X)\xi = \frac{a+b}{2n}\{\eta(X)\xi - X\} \tag{12}$$

C. *Proposition*

In an *(2n+1)*-dimensional N(k)-quasi Einstein manifold, the W_2 curvature tensor satisfies

$$W_2(X,Y)\xi = \frac{b}{2n}(\eta(Y)X - \eta(X)Y) \tag{13}$$

$$W_2(\xi,X)\xi = \frac{b}{2n}(Y - \eta(Y)\xi) = -W_2(X,\xi)\xi \tag{14}$$

$$\eta(W_2(X,Y)\xi) = 0 \tag{15}$$

$$W_2(\xi,Y)Z = -\frac{b}{2n}(Y - \eta(Y)\xi)\eta(Z) \tag{16}$$

$$\eta(W_2(\xi,Y)Z) = 0 \tag{17}$$

Proof - where N(k) - quasi

D. *Theorem*

Let M be *(2n+1)* dimensional *N(k)* quasi Einstein manifold. If M satisfies the condition $W_2(\xi,X).W_2 = 0$ then M is Einstein and $a = 0$.

Proof:

Let $W_2(\xi,X)\cdot W_2 = 0$, this implies

$$0 = [W_2(\xi,U),W_2(X,Y)]\xi - W_2(W_2(\xi,U)X,Y)\xi - W_2(X,W_2(\xi,U)Y)\xi$$

$$0 = W_2(\xi,U)W_2(X,Y)\xi - W_2(X,Y)W_2(\xi,U)\xi - W_2(W_2(\xi,U)X,Y)\xi - W_2(X,W_2(\xi,U)Y)\xi$$

Using equations (14), (15) and (16) in above equation we get

$$W_2(X,Y)Z = \frac{b}{2n}(\eta(Y)X - \eta(X)Y)\eta(Z).$$

Using (4) in above equation we get

$$R(X,Y,Z,W) = \frac{b}{2n}(g(X,W)\eta(Z) - g(Y,W)\eta(X)\eta(Z)) + \frac{1}{2n}(g(Y,Z)S(X,W) - g(X,Z)S(Y,W))$$

Contracting above equation and using eq. (1) and (8) we get

$$\frac{b}{2n}(g(Y,Z) - \eta(Y)\eta(Z)) = 0$$

But $b \neq 0$, we get

$$g(Y,Z) = \eta(Y)\eta(Z)$$

Using above equation in (1) we get

$$S(X,Y) = (a+b)g(X,Y).$$

This equation shows that manifold is Einstein. Contacting above equation and using (8) we get

a = 0

Corollary:

Let *M* be *(2n + 1)* dimensional *N(k)* quasi Einstein manifold. If *M* satisfies the condition $W_2(\xi, X)\cdot W_2 = 0$.

Then

$$R(X,Y)Z = \frac{b}{2n}\{\eta(Y)X - \eta(X)Y\}\eta(Z),$$

$$X, Y, Z \in TM$$

III. CONCLUSION

A (2n + 1)-dimensional N(k) - quasi Einstein manifold M is Eistein if it satisfies the conditions of $W_2(\xi, X).W_2 = 0$.

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